

Noether's Theorem for Nonsmooth Extremals of Variational Problems with Time Delay*

Gastão S. F. Frederico^{1,2}
gastao.frederico@ua.pt

Tatiana Odziejewicz¹
tatianaod@ua.pt

Delfim F. M. Torres¹
delfim@ua.pt

¹CIDMA — Center for Research and Development in Mathematics and Applications,
Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

²Department of Science and Technology,
University of Cape Verde, Praia, Santiago, Cape Verde

Abstract

We obtain a nonsmooth extension of Noether's symmetry theorem for variational problems with delayed arguments. The result is proved to be valid in the class of Lipschitz functions, as long as the delayed Euler–Lagrange extremals are restricted to those that satisfy the DuBois–Reymond necessary optimality condition. The important case of delayed variational problems with higher-order derivatives is considered as well.

Keywords: time delays; invariance; symmetries; constants of motion; conservation laws; DuBois–Reymond necessary optimality condition; Noether's theorem.

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1 Introduction

In 1918, Emmy Noether published a paper that strongly influenced the physics of the 20th century [14]. She proved a theorem asserting that if the Lagrangian is invariant under changes in the coordinate system, then there exists a conserved quantity along all the Euler–Lagrange extremals. Within the years, this result has been studied by many authors and generalized in different directions: see [1, 4–6, 9, 15, 16, 18] and references therein. In particular, in the recent paper [7], Noether's theorem was formulated for variational problems with delayed arguments. The result is important because problems with delays play a crucial role in the modeling of real-life phenomena in various fields of applications [8]. In order to prove Noether's theorem with delays, it was assumed that admissible functions are \mathcal{C}^2 -smooth and that Noether's conserved quantity holds along all \mathcal{C}^2 -extremals of the Euler–Lagrange equations with time delay [7]. Here we remark that when one extends Noether's theorem to the biggest class for which one can derive the Euler–Lagrange equations, i.e., for Lipschitz continuous functions, then one can find Lipschitz Euler–Lagrange extremals that fail to satisfy the Noether conserved quantity established in [7] (see a simple example in Section 3). We show that to formulate Noether's theorem with time delays for nonsmooth functions, it is enough to restrict the set of delayed Euler–Lagrange extremals to those that satisfy the delayed DuBois–Reymond condition. Moreover, we prove that this result can be generalized to higher-order variational problems.

The text is organized as follows. In Section 2 the fundamental problem of variational calculus with delayed arguments is formulated and a short review of the results for \mathcal{C}^2 -smooth admissible

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functions is given. In Section 3 we show, through an example, that nonsmooth Euler–Lagrange delayed extremals may fail to satisfy Noether’s constants of motion [7]. The main contributions of the paper appear in Sections 4 and 5: we prove a Noether symmetry theorem with time delay for Lipschitz functions (Theorem 11), Euler–Lagrange and DuBois–Reymond optimality type conditions for nonsmooth higher-order variational problems with delayed arguments (Theorems 15 and 18, respectively), and a delayed higher-order Noether’s symmetry theorem (Theorem 24).

2 Preliminaries

In this section we review necessary results on the calculus of variations with time delay. For more on variational problems with delayed arguments we refer the reader to [2, 3, 8, 10, 12, 13, 17].

The fundamental problem consists of minimizing a functional

$$J^\tau[q(\cdot)] = \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t), q(t-\tau), \dot{q}(t-\tau)) dt \quad (1)$$

subject to boundary conditions

$$q(t) = \delta(t) \text{ for } t \in [t_1 - \tau, t_1] \text{ and } q(t_2) = q_{t_2}. \quad (2)$$

We assume that the Lagrangian $L : [t_1, t_2] \times \mathbb{R}^{4n} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, is a \mathcal{C}^2 -function with respect to all its arguments, admissible functions $q(\cdot)$ are \mathcal{C}^2 -smooth, $t_1 < t_2$ are fixed in \mathbb{R} , τ is a given positive real number such that $\tau < t_2 - t_1$, and δ is a given piecewise smooth function on $[t_1 - \tau, t_1]$. Throughout the text, $\partial_i L$ denotes the partial derivative of L with respect to its i th argument, $i = 1, \dots, 5$. For convenience of notation, we introduce the operator $[\cdot]_\tau$ defined by

$$[q]_\tau(t) = (t, q(t), \dot{q}(t), q(t-\tau), \dot{q}(t-\tau)).$$

The next theorem gives a necessary optimality condition of Euler–Lagrange type for (1)–(2).

Theorem 1 (Euler–Lagrange equations with time delay [10]). *If $q(\cdot) \in \mathcal{C}^2$ is a minimizer for problem (1)–(2), then $q(\cdot)$ satisfies the following Euler–Lagrange equations with time delay:*

$$\begin{cases} \frac{d}{dt} \{ \partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t+\tau) \} = \partial_2 L[q]_\tau(t) + \partial_4 L[q]_\tau(t+\tau), & t_1 \leq t \leq t_2 - \tau, \\ \frac{d}{dt} \partial_3 L[q]_\tau(t) = \partial_2 L[q]_\tau(t), & t_2 - \tau \leq t \leq t_2. \end{cases} \quad (3)$$

Remark 2. *If one extends the set of admissible functions in problem (1)–(2) to the class of Lipschitz continuous functions, then the Euler–Lagrange equations (3) remain valid. This result is obtained from our Corollary 16 by choosing $m = 1$.*

Definition 3 (Extremals). *The solutions $q(\cdot)$ of the Euler–Lagrange equations (3) with time delay are called extremals.*

Definition 4 (Invariance of (1)). *Consider the following s -parameter group of infinitesimal transformations:*

$$\begin{cases} \bar{t} = t + s\eta(t, q) + o(s), \\ \bar{q}(t) = q(t) + s\xi(t, q) + o(s), \end{cases} \quad (4)$$

where $\eta \in \mathcal{C}^1(\mathbb{R}^{n+1}, \mathbb{R})$ and $\xi \in \mathcal{C}^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$. Functional (1) is said to be invariant under (4) if

$$0 = \frac{d}{ds} \int_{\bar{I}(I)} L \left(t + s\eta(t, q(t)) + o(s), q(t) + s\xi(t, q(t)) + o(s), \frac{\dot{q}(t) + s\dot{\xi}(t, q(t))}{1 + s\dot{\eta}(t, q(t))}, \right. \\ \left. q(t-\tau) + s\xi(t-\tau, q(t-\tau)) + o(s), \frac{\dot{q}(t-\tau) + s\dot{\xi}(t-\tau, q(t-\tau))}{1 + s\dot{\eta}(t-\tau, q(t-\tau))} \right) (1 + s\dot{\eta}(t, q(t))) dt \Big|_{s=0}$$

for any subinterval $I \subseteq [t_1, t_2]$.

Definition 5 (Constant of motion/conservation law with time delay). *We say that a quantity $C(t, t + \tau, q(t), q(t - \tau), q(t + \tau), \dot{q}(t), \dot{q}(t - \tau), \dot{q}(t + \tau))$ is a constant of motion with time delay τ if*

$$\frac{d}{dt} C(t, t + \tau, q(t), q(t - \tau), q(t + \tau), \dot{q}(t), \dot{q}(t - \tau), \dot{q}(t + \tau)) = 0 \quad (5)$$

along all the extremals $q(\cdot)$ (cf. Definition 3). The equality (5) is then a conservation law with time delay.

Next theorem extends the DuBois–Reymond necessary optimality condition to problems of the calculus of variations with time delay.

Theorem 6 (DuBois–Reymond necessary conditions with time delay [7]). *If $q(\cdot) \in \mathcal{C}^2$ is an extremal of functional (1) subject to (2), then the following conditions are satisfied:*

$$\begin{cases} \frac{d}{dt} \{L[q]_\tau(t) - \dot{q}(t) \cdot (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau))\} = \partial_1 L[q]_\tau(t), & t_1 \leq t \leq t_2 - \tau, \\ \frac{d}{dt} \{L[q]_\tau(t) - \dot{q}(t) \cdot \partial_3 L[q]_\tau(t)\} = \partial_1 L[q]_\tau(t), & t_2 - \tau \leq t \leq t_2. \end{cases} \quad (6)$$

Remark 7. *If we assume that admissible functions in problem (1)–(2) are Lipschitz continuous, then one can show that the DuBois–Reymond necessary conditions with time delay (6) are still valid (cf. Corollary 19).*

Theorem 8 establishes an extension of Noether’s theorem to problems of the calculus of variations with time delay.

Theorem 8 (Noether’s symmetry theorem with time delay [7]). *If functional (1) is invariant in the sense of Definition 4, then the quantity $C(t, t + \tau, q(t), q(t - \tau), q(t + \tau), \dot{q}(t), \dot{q}(t - \tau), \dot{q}(t + \tau))$ defined by*

$$\begin{aligned} & (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \cdot \xi(t, q(t)) \\ & + \left(L[q]_\tau(t) - \dot{q}(t) \cdot (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \right) \eta(t, q(t)) \end{aligned} \quad (7)$$

for $t_1 \leq t \leq t_2 - \tau$ and by

$$\partial_3 L[q]_\tau(t) \cdot \xi(t, q(t)) + \left(L[q]_\tau(t) - \dot{q}(t) \cdot \partial_3 L[q]_\tau(t) \right) \eta(t, q(t)) \quad (8)$$

for $t_2 - \tau \leq t \leq t_2$, is a constant of motion with time delay (cf. Definition 5).

3 Nonsmooth Euler–Lagrange extremals may fail to satisfy Noether’s conservation laws with time delay

Consider the problem of the calculus of variations with time delay

$$\begin{aligned} J^1[q(\cdot)] &= \int_0^3 (\dot{q}(t) + \dot{q}(t - 1))^2 dt \longrightarrow \min, \\ q(t) &= -t, \quad -1 \leq t \leq 0, \quad q(3) = 1, \end{aligned} \quad (9)$$

in the class of functions $q(\cdot) \in Lip([-1, 3]; \mathbb{R})$. From Theorem 1 (see Remark 2), one obtains that any solution to problem (9) must satisfy

$$2\dot{q}(t) + \dot{q}(t - 1) + \dot{q}(t + 1) = c_1, \quad 0 \leq t \leq 2, \quad (10)$$

$$\dot{q}(t) + \dot{q}(t - 1) = c_2, \quad 2 \leq t \leq 3, \quad (11)$$

where c_1 and c_2 are constants. Because functional J^1 of problem (9) is autonomous, we have invariance, in the sense of Definition 4, with $\eta \equiv 1$ and $\xi \equiv 0$. Simple calculations show that Noether's constant of motion with time delay (7)–(8) coincides with the DuBois–Reymond condition (6):

$$(\dot{q}(t) + \dot{q}(t-1))^2 - 2\dot{q}(t)(2\dot{q}(t) + \dot{q}(t-1) + \dot{q}(t+1)) = c_3, \quad 0 \leq t \leq 2, \quad (12)$$

$$\dot{q}(t)^2 - \dot{q}(t-1)^2 = c_4, \quad 2 \leq t \leq 3, \quad (13)$$

where c_3 and c_4 are constants. One can easily check that function

$$q(t) = \begin{cases} -t & \text{for } -1 < t \leq 0 \\ t & \text{for } 0 < t \leq 2 \\ -t + 4 & \text{for } 2 < t \leq 3 \end{cases} \quad (14)$$

satisfies (10)–(11) with $c_1 = 2$ and $c_2 = 0$, but does not satisfy (12)–(13): for $0 < t \leq 1$ constant c_3 should be -4 and for $1 < t \leq 2$ constant c_3 should be 0 . We conclude that nonsmooth solutions of Euler–Lagrange equations (3) do not preserve Noether's quantity defined by (7)–(8) and one needs to restrict the set of Euler–Lagrange extremals. In Section 4 we show that it is enough to restrict the Euler–Lagrange extremals to those that satisfy the DuBois–Reymond necessary condition (6).

4 Noether's theorem with time delay for Lipschitz functions

The notion of invariance given in Definition 4 can be extended up to an exact differential.

Definition 9 (Invariance up to a gauge-term). *We say that functional (1) is invariant under the s -parameter group of infinitesimal transformations (4) up to the gauge-term Φ if*

$$\int_I \dot{\Phi}[q]_\tau(t) dt = \frac{d}{ds} \int_{\bar{t}(I)} L \left(t + s\eta(t, q(t)) + o(s), q(t) + s\xi(t, q(t)) + o(s), \frac{\dot{q}(t) + s\dot{\xi}(t, q(t))}{1 + s\dot{\eta}(t, q(t))}, \right. \\ \left. q(t - \tau) + s\xi(t - \tau, q(t - \tau)) + o(s), \frac{\dot{q}(t - \tau) + s\dot{\xi}(t - \tau, q(t - \tau))}{1 + s\dot{\eta}(t - \tau, q(t - \tau))} \right) (1 + s\dot{\eta}(t, q(t))) dt \Big|_{s=0} \quad (15)$$

for any subinterval $I \subseteq [t_1, t_2]$ and for all $q(\cdot) \in \text{Lip}([t_1 - \tau, t_2]; \mathbb{R}^n)$.

Lemma 10 (Necessary condition of invariance). *If functional (1) is invariant up to Φ in the sense of Definition 9, then*

$$\int_{t_1}^{t_2 - \tau} \left[-\dot{\Phi}[q]_\tau(t) + \partial_1 L[q]_\tau(t) \eta(t, q) + (\partial_2 L[q]_\tau(t) + \partial_4 L[q]_\tau(t + \tau)) \cdot \xi(t, q) \right. \\ \left. + (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \cdot \left(\dot{\xi}(t, q) - \dot{q}(t) \dot{\eta}(t, q) \right) + L[q]_\tau(t) \dot{\eta}(t, q) \right] dt = 0 \quad (16)$$

for $t_1 \leq t \leq t_2 - \tau$ and

$$\int_{t_2 - \tau}^{t_2} \left[-\dot{\Phi}[q]_\tau(t) + \partial_1 L[q]_\tau(t) \eta(t, q) + \partial_2 L[q]_\tau(t) \cdot \xi(t, q) \right. \\ \left. + \partial_3 L[q]_\tau(t) \cdot \left(\dot{\xi}(t, q) - \dot{q}(t) \dot{\eta}(t, q) \right) + L[q]_\tau(t) \dot{\eta}(t, q) \right] dt = 0 \quad (17)$$

for $t_2 - \tau \leq t \leq t_2$.

Proof. Without loss of generality, we take $I = [t_1, t_2]$. Then, (15) is equivalent to

$$\begin{aligned} & \int_{t_1}^{t_2} \left[-\dot{\Phi}[q]_\tau(t) + \partial_1 L[q]_\tau(t) \eta(t, q) + \partial_2 L[q]_\tau(t) \cdot \xi(t, q) \right. \\ & \quad \left. + \partial_3 L[q]_\tau(t) \cdot \left(\dot{\xi}(t, q) - \dot{q}(t) \dot{\eta}(t, q) \right) + L[q]_\tau(t) \dot{\eta}(t, q) \right] dt \\ & \quad + \int_{t_1}^{t_2} \left[\partial_4 L[q]_\tau(t) \cdot \xi(t - \tau, q(t - \tau)) \right. \\ & \quad \left. + \partial_5 L[q]_\tau(t) \cdot \left(\dot{\xi}(t - \tau, q(t - \tau)) - \dot{q}(t - \tau) \dot{\eta}(t - \tau, q(t - \tau)) \right) \right] dt = 0. \end{aligned} \quad (18)$$

Performing a linear change of variables $t = \sigma + \tau$ in the last integral of (18), and keeping in mind that $L[q]_\tau(t) \equiv 0$ on $[t_1 - \tau, t_1]$, equation (18) becomes

$$\begin{aligned} & \int_{t_1}^{t_2 - \tau} \left[-\dot{\Phi}[q]_\tau(t) + \partial_1 L[q]_\tau(t) \eta(t, q) + (\partial_2 L[q]_\tau(t) + \partial_4 L[q]_\tau(t + \tau)) \cdot \xi(t, q) \right. \\ & \quad \left. + (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \cdot \left(\dot{\xi}(t, q) - \dot{q}(t) \dot{\eta}(t, q) \right) + L[q]_\tau(t) \dot{\eta}(t, q) \right] dt \\ & \quad + \int_{t_2 - \tau}^{t_2} \left[-\dot{\Phi}[q]_\tau(t) + \partial_1 L[q]_\tau(t) \eta(t, q) + \partial_2 L[q]_\tau(t) \cdot \xi(t, q) \right. \\ & \quad \left. + \partial_3 L[q]_\tau(t) \cdot \left(\dot{\xi}(t, q) - \dot{q}(t) \dot{\eta}(t, q) \right) + L[q]_\tau(t) \dot{\eta}(t, q) \right] dt = 0. \end{aligned} \quad (19)$$

Taking into consideration that (19) holds for an arbitrary subinterval $I \subseteq [t_1, t_2]$, equations (16) and (17) hold. \square

Theorem 11 (Noether's symmetry theorem with time delay for Lipschitz functions). *If functional (1) is invariant up to Φ in the sense of Definition 9, then the quantity $C(t, t + \tau, q(t), q(t - \tau), q(t + \tau), \dot{q}(t), \dot{q}(t - \tau), \dot{q}(t + \tau))$ defined by*

$$\begin{aligned} & -\Phi[q]_\tau(t) + (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \cdot \xi(t, q(t)) \\ & \quad + \left(L[q]_\tau - \dot{q}(t) \cdot (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \right) \eta(t, q(t)) \end{aligned} \quad (20)$$

for $t_1 \leq t \leq t_2 - \tau$ and by

$$-\Phi[q]_\tau(t) + \partial_3 L[q]_\tau(t) \cdot \xi(t, q(t)) + \left(L[q]_\tau - \dot{q}(t) \cdot \partial_3 L[q]_\tau(t) \right) \eta(t, q(t)) \quad (21)$$

for $t_2 - \tau \leq t \leq t_2$, is a constant of motion with time delay along any $q(\cdot) \in \text{Lip}([t_1 - \tau, t_2]; \mathbb{R}^n)$ satisfying both (3) and (6), i.e., along any Lipschitz Euler–Lagrange extremal that is also a Lipschitz DuBois–Reymond extremal.

Proof. We prove the theorem in the interval $t_1 \leq t \leq t_2 - \tau$. The proof is similar for the interval $t_2 - \tau \leq t \leq t_2$. Noether's constant of motion with time delay (20) follows by using in the interval $t_1 \leq t \leq t_2 - \tau$ the DuBois–Reymond condition with time delay (6) and the Euler–Lagrange equation with time delay (3) into the necessary condition of invariance (16):

$$\begin{aligned} 0 &= \int_{t_1}^{t_2 - \tau} \left[-\dot{\Phi}[q]_\tau(t) + \partial_1 L[q]_\tau(t) \eta(t, q) + (\partial_2 L[q]_\tau(t) + \partial_4 L[q]_\tau(t + \tau)) \cdot \xi(t, q) \right. \\ & \quad \left. + (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \cdot \left(\dot{\xi}(t, q) - \dot{q}(t) \dot{\eta}(t, q) \right) + L[q]_\tau(t) \dot{\eta}(t, q) \right] dt \\ &= \int_{t_1}^{t_2 - \tau} \left[\frac{d}{dt} \left\{ -\Phi[q]_\tau(t) + (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \cdot \xi(t, q) \right. \right. \\ & \quad \left. \left. + (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \cdot \dot{\xi}(t, q) \right. \right. \\ & \quad \left. \left. + \frac{d}{dt} \{ L[q]_\tau(t) - \dot{q}(t) \cdot (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \} \eta(t, q) \right. \right. \\ & \quad \left. \left. + \{ L[q]_\tau(t) - \dot{q}(t) \cdot (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \} \dot{\eta}(t, q) \right\} \right] dt, \end{aligned}$$

that is,

$$\int_{t_1}^{t_2-\tau} \frac{d}{dt} \left[-\Phi[q]_\tau(t) + (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t+\tau)) \cdot \xi(t, q(t)) \right. \\ \left. + \left(L[q]_\tau(t) - \dot{q}(t) \cdot (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t+\tau)) \right) \eta(t, q(t)) \right] dt = 0. \quad (22)$$

Taking into consideration that (22) holds for any subinterval $I \subseteq [t_1, t_2]$, we conclude that

$$-\Phi[q]_\tau(t) + (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t+\tau)) \cdot \xi(t, q(t)) \\ + \left(L[q]_\tau(t) - \dot{q}(t) \cdot (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t+\tau)) \right) \eta(t, q(t)) = \text{constant}.$$

□

Example 12. Consider problem (9). Function $q(\cdot) \in \text{Lip}([-1, 3]; \mathbb{R}^n)$ defined by

$$q(t) = \begin{cases} -t & \text{for } -1 < t \leq 0 \\ t & \text{for } 0 < t \leq 1 \\ -t+2 & \text{for } 1 < t \leq 2 \\ t-2 & \text{for } 2 < t \leq 3 \end{cases} \quad (23)$$

is an Euler–Lagrange extremal, i.e., satisfies (10)–(11), but, in contrast with (14), is also a DuBois–Reymond extremal, i.e., satisfies (12)–(13). Theorem 11 asserts the validity of Noether’s constant of motion, which is here easily verified: (20)–(21) holds along (23) with $\Phi \equiv 0$, $\eta \equiv 1$, and $\xi \equiv 0$.

5 Nonsmooth higher-order Noether’s theorem for problems of the calculus of variations with time delay

Let $\mathbb{W}^{k,p}$, $k \geq 1$, $1 \leq p \leq \infty$, denote the class of functions that are absolutely continuous with their derivatives up to order $k-1$, the k th derivative belonging to L^p . With this notation, the class Lip of Lipschitz functions is represented by $\mathbb{W}^{1,\infty}$. We now extend previous results to problems with higher-order derivatives.

5.1 Higher-order Euler–Lagrange and DuBois–Reymond optimality conditions with time delay

Let $m \in \mathbb{N}$ and $q^{(i)}(t)$ denote the i th derivative of $q(t)$, $i = 0, \dots, m$, with $q^{(0)}(t) = q(t)$. For simplicity of notation, we introduce the operator $[\cdot]_\tau^m$ by

$$[q]_\tau^m(t) := \left(t, q(t), \dot{q}(t), \dots, q^{(m)}(t), q(t-\tau), \dot{q}(t-\tau), \dots, q^{(m)}(t-\tau) \right).$$

Consider the following higher-order variational problem with time delay: to minimize

$$J_m^\tau[q(\cdot)] = \int_{t_1}^{t_2} L[q]_\tau^m(t) dt \quad (24)$$

subject to the boundary conditions (2) and $q^{(i)}(t_2) = q_{t_2}^i$, $i = 1, \dots, m-1$. The Lagrangian $L : [t_1, t_2] \times \mathbb{R}^{2n(m+1)} \rightarrow \mathbb{R}$ is assumed to be a \mathcal{C}^{m+1} -function with respect to all its arguments, admissible functions $q(\cdot)$ are assumed to be $\mathbb{W}^{m,\infty}$, $t_1 < t_2$ are fixed in \mathbb{R} , τ is a given positive real number such that $\tau < t_2 - t_1$, and $q_{t_2}^i$ are given vectors in \mathbb{R}^n , $i = 1, \dots, m-1$.

Remark 13. When $m = 1$ functional (24) reduces to (1), i.e., $J_1^\tau = J^\tau$.

A variation of $q \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$ is another function in the set $\mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$ of the form $q + \varepsilon h$, with $h \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$, such that $h^{(i)}(t_2) = 0$, $i = 0, \dots, m$, $h(t) = 0$ if $t \in [t_1 - \tau, t_1]$, and ε a small real positive number.

Definition 14 (Extremal of (24)). *We say that q is an extremal of the delayed functional (24) if for any $h(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$ such that $h^{(i)}(t_2) = 0$, $i = 0, \dots, m-1$, and $h(t) = 0$, $t \in [t_1 - \tau, t_1]$, the following equation holds:*

$$\frac{d}{d\varepsilon} J_m^\tau[q + \varepsilon h]|_{\varepsilon=0} = 0.$$

Theorem 15 (Higher-order Euler–Lagrange equations with time delay in integral form). *If $q(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$ is an extremal of functional (24), then $q(\cdot)$ satisfies the following higher-order Euler–Lagrange integral equations with time delay:*

$$\sum_{i=0}^m (-1)^{m-i-1} \left(\underbrace{\int_{t_2-\tau}^t \int_{t_2-\tau}^{s_1} \dots \int_{t_2-\tau}^{s_{m-i-1}}}_{m-i \text{ times}} \left(\partial_{i+2} L[q]_\tau^m(s_{m-i}) \right. \right. \\ \left. \left. + \partial_{i+m+3} L[q]_\tau^m(s_{m-i} + \tau) \right) ds_{m-i} \dots ds_2 ds_1 \right) = p(t) \quad (25)$$

for $t_1 \leq t \leq t_2 - \tau$ and

$$\sum_{i=0}^m (-1)^{m-i-1} \left(\underbrace{\int_{t_2-\tau}^t \int_{t_2-\tau}^{s_1} \dots \int_{t_2-\tau}^{s_{m-i-1}}}_{m-i \text{ times}} \left(\partial_{i+2} L[q]_\tau^m(t) \right) ds_{m-i} \dots ds_2 ds_1 \right) = p(t) \quad (26)$$

for $t_2 - \tau \leq t \leq t_2$, where $p(t)$ is a polynomial of order $m-1$, i.e., $p(t) = c_0 + c_1 t + \dots + c_{m-1} t^{m-1}$ for some constants $c_i \in \mathbb{R}$, $i = 0, \dots, m-1$.

Proof. Assume that $q(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$ is an extremal of functional (24). According to the Definition 14, for any $h(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$ such that $h^{(i)}(t_2) = 0$, $i = 0, \dots, m-1$, and $h(t) = 0$, $t \in [t_1 - \tau, t_1]$, we have

$$\int_{t_1}^{t_2} \left(\sum_{i=0}^m \partial_{i+2} L[q]_\tau^m(t) \cdot h^{(i)}(t) + \sum_{i=0}^m \partial_{i+m+3} L[q]_\tau^m(t) \cdot h^{(i)}(t - \tau) \right) dt = 0. \quad (27)$$

Performing the linear change of variables $t = \sigma + \tau$ in the last term of integral (27), and using the fact that $h(t) = 0$ if $t \in [t_1 - \tau, t_1]$, (27) becomes

$$\int_{t_1}^{t_2} \left(\sum_{i=0}^m \partial_{i+2} L[q]_\tau^m(t) \cdot h^{(i)}(t) \right) dt + \int_{t_1}^{t_2-\tau} \left(\sum_{i=0}^m \partial_{i+m+3} L[q]_\tau^m(t + \tau) \cdot h^{(i)}(t) \right) dt = 0. \quad (28)$$

By repeated integration by parts one has

$$\sum_{i=0}^m \int_{t_1}^{t_2} \partial_{i+2} L[q]_\tau^m(t) \cdot h^{(i)}(t) dt \\ = \sum_{i=0}^m \left\{ \left[\sum_{j=1}^{m-i} (-1)^{j+1} h^{(i+j-1)}(t) \cdot \left(\underbrace{\int_{t_2-\tau}^t \int_{t_2-\tau}^{s_1} \dots \int_{t_2-\tau}^{s_{j-1}}}_{j \text{ times}} \left(\partial_{i+2} L[q]_\tau^m(s_j) \right) ds_j \dots ds_2 ds_1 \right) \right]_{t_1}^{t_2} \right. \\ \left. + (-1)^i \int_{t_1}^{t_2} h^{(m)}(t) \cdot \left(\underbrace{\int_{t_2-\tau}^t \int_{t_2-\tau}^{s_1} \dots \int_{t_2-\tau}^{s_{m-i-1}}}_{m-i \text{ times}} \left(\partial_{i+2} L[q]_\tau^m(s_{m-i}) \right) ds_{m-i} \dots ds_2 ds_1 \right) dt \right\} \quad (29)$$

and

$$\begin{aligned}
& \sum_{i=0}^m \int_{t_1}^{t_2-\tau} \partial_{i+m+3} L[q]_{\tau}^m(t+\tau) \cdot h^{(i)}(t) dt \\
&= \sum_{i=0}^m \left\{ \left[\sum_{j=1}^{m-i} (-1)^{j+1} h^{(i+j-1)}(t) \cdot \underbrace{\left(\int_{t_2-\tau}^t \int_{t_2-\tau}^{s_1} \cdots \int_{t_2-\tau}^{s_{j-1}} \left(\partial_{i+m+3} L[q]_{\tau}^m(s_j+\tau) \right) ds_j \cdots ds_2 ds_1 \right)}_{j \text{ times}} \right]_{t_1}^{t_2-\tau} \right. \\
& \quad \left. + (-1)^i \int_{t_1}^{t_2-\tau} h^{(m)}(t) \cdot \underbrace{\left(\int_{t_2-\tau}^t \int_{t_2-\tau}^{s_1} \cdots \int_{t_2-\tau}^{s_{m-i-1}} \left(\partial_{i+m+3} L[q]_{\tau}^m(s_{m-i}+\tau) \right) ds_{m-i} \cdots ds_2 ds_1 \right)}_{m-i \text{ times}} dt \right\}.
\end{aligned} \tag{30}$$

Because $h^{(i)}(t_2) = 0$, $i = 0, \dots, m-1$, and $h(t) = 0$, $t \in [t_1 - \tau, t_1]$, the terms without integral sign in the right-hand sides of identities (29) and (30) vanish. Therefore, equation (28) becomes

$$\begin{aligned}
0 &= \int_{t_1}^{t_2-\tau} h^{(m)}(t) \cdot \left[\sum_{i=0}^m (-1)^i \underbrace{\left(\int_{t_2-\tau}^t \int_{t_2-\tau}^{s_1} \cdots \int_{t_2-\tau}^{s_{m-i-1}} \left(\partial_{i+2} L[q]_{\tau}^m(s_{m-i}) \right. \right. \right. \\
& \quad \left. \left. \left. + \partial_{i+m+3} L[q]_{\tau}^m(s_{m-i} + \tau) \right) ds_{m-i} \cdots ds_2 ds_1 \right)}_{m-i \text{ times}} \right] dt \\
&+ \int_{t_2-\tau}^{t_2} h^{(m)}(t) \cdot \left[\sum_{i=0}^m (-1)^i \underbrace{\left(\int_{t_2-\tau}^t \int_{t_2-\tau}^{s_1} \cdots \int_{t_2-\tau}^{s_{m-i-1}} \left(\partial_{i+2} L[q]_{\tau}^m(s_{m-i}) \right) ds_{m-i} \cdots ds_2 ds_1 \right)}_{m-i \text{ times}} \right] dt.
\end{aligned} \tag{31}$$

For $i = 0, \dots, m$ we define functions

$$\varphi_i(t) = \begin{cases} \partial_{i+2} L[q]_{\tau}^m(t) + \partial_{i+m+3} L[q]_{\tau}^m(t+\tau) & \text{for } t_1 \leq t \leq t_2 - \tau \\ \partial_{i+2} L[q]_{\tau}^m(t) & \text{for } t_2 - \tau \leq t \leq t_2. \end{cases}$$

Then one can write equation (31) as follows:

$$0 = \int_{t_1}^{t_2} h^{(m)}(t) \cdot \left[\sum_{i=0}^m (-1)^i \underbrace{\left(\int_{t_2-\tau}^t \int_{t_2-\tau}^{s_1} \cdots \int_{t_2-\tau}^{s_{m-i-1}} \left(\varphi_i(s_{m-i}) \right) ds_{m-i} \cdots ds_2 ds_1 \right)}_{m-i \text{ times}} \right] dt.$$

Applying the higher-order DuBois–Reymond lemma [11, 20], one arrives to (25) and (26). \square

Corollary 16 (Higher-order Euler–Lagrange equations with time delay in differential form). *If $q(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$ is an extremal of functional (24), then*

$$\sum_{i=0}^m (-1)^i \frac{d^i}{dt^i} \left(\partial_{i+2} L[q]_{\tau}^m(t) + \partial_{i+m+3} L[q]_{\tau}^m(t+\tau) \right) = 0 \tag{32}$$

for $t_1 \leq t \leq t_2 - \tau$ and

$$\sum_{i=0}^m (-1)^i \frac{d^i}{dt^i} \partial_{i+2} L[q]_{\tau}^m(t) = 0 \tag{33}$$

for $t_2 - \tau \leq t \leq t_2$.

Proof. We obtain (32) and (33) applying the derivative of order m to (25) and (26), respectively. \square

Remark 17. If $m = 1$, then the higher-order Euler–Lagrange equations (32)–(33) reduce to (3).

Associated to a given function $q(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$, it is convenient to introduce the following quantities (cf. [19]):

$$\psi_1^j = \sum_{i=0}^{m-j} (-1)^i \frac{d^i}{dt^i} \left(\partial_{i+j+2} L[q]_\tau^m(t) + \partial_{i+j+m+3} L[q]_\tau^m(t + \tau) \right) \quad (34)$$

for $t_1 \leq t \leq t_2 - \tau$, and

$$\psi_2^j = \sum_{i=0}^{m-j} (-1)^i \frac{d^i}{dt^i} \partial_{i+j+2} L[q]_\tau^m(t) \quad (35)$$

for $t_2 - \tau \leq t \leq t_2$, where $j = 0, \dots, m$. These operators are useful for our purposes because of the following properties:

$$\frac{d}{dt} \psi_1^j = \partial_{j+1} L[q]_\tau^m(t) + \partial_{j+m+2} L[q]_\tau^m(t + \tau) - \psi_1^{j-1} \quad (36)$$

for $t_1 \leq t \leq t_2 - \tau$, and

$$\frac{d}{dt} \psi_2^j = \partial_{j+1} L[q]_\tau^m(t) - \psi_2^{j-1}$$

for $t_2 - \tau \leq t \leq t_2$, where $j = 1, \dots, m$. We are now in conditions to prove a higher-order DuBois–Reymond optimality condition for problems with time delay.

Theorem 18 (Higher-order delayed DuBois–Reymond condition). *If $q(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$ is an extremal of functional (24), then*

$$\frac{d}{dt} \left(L[q]_\tau^m(t) - \sum_{j=1}^m \psi_1^j \cdot q^{(j)}(t) \right) = \partial_1 L[q]_\tau^m(t) \quad (37)$$

for $t_1 \leq t \leq t_2 - \tau$ and

$$\frac{d}{dt} \left(L[q]_\tau^m(t) - \sum_{j=1}^m \psi_2^j \cdot q^{(j)}(t) \right) = \partial_1 L[q]_\tau^m(t) \quad (38)$$

for $t_2 - \tau \leq t \leq t_2$, where ψ_1^j is given by (34) and ψ_2^j by (35).

Proof. We prove the theorem in the interval $t_1 \leq t \leq t_2 - \tau$. The proof is similar for $t_2 - \tau \leq t \leq t_2$. We derive equation (37) as follows:

$$\begin{aligned} & \int_{t_1}^{t_2} \frac{d}{dt} \left(L[q]_\tau^m(t) - \sum_{j=1}^m \psi_1^j \cdot q^{(j)}(t) \right) dt \\ &= \int_{t_1}^{t_2} \left(\partial_1 L[q]_\tau^m(t) + \sum_{j=0}^m \partial_{j+2} L[q]_\tau^m(t) \cdot q^{(j+1)}(t) - \sum_{j=1}^m \left(\psi_1^j \cdot q^{(j)}(t) + \psi_1^j \cdot q^{(j+1)}(t) \right) \right) dt \\ & \quad + \int_{t_1}^{t_2} \sum_{j=0}^m \partial_{j+m+3} L[q]_\tau^m(t) \cdot q^{(j+1)}(t - \tau) dt. \quad (39) \end{aligned}$$

From (36) and by performing a linear change of variables $t = \sigma + \tau$ in the last integral of (39), in the interval where $t_1 \leq t \leq t_2 - \tau$, the equation (39) becomes

$$\begin{aligned} \int_{t_1}^{t_2-\tau} \frac{d}{dt} \left(L[q]_\tau^m(t) - \sum_{j=1}^m \psi_1^j \cdot q^{(j)}(t) \right) dt &= \int_{t_1}^{t_2-\tau} \left[\partial_1 L[q]_\tau^m(t) + \sum_{j=0}^m \partial_{j+2} L[q]_\tau^m(t) \cdot q^{(j+1)}(t) \right. \\ &\quad \left. - \sum_{j=1}^m \left((\partial_{j+1} L[q]_\tau^m(t) + \partial_{j+m+2} L[q]_\tau^m(t+\tau) - \psi_1^{j-1}) \cdot q^{(j)}(t) + \psi_1^j \cdot q^{(j+1)}(t) \right) \right. \\ &\quad \left. + \sum_{j=0}^m \partial_{j+m+3} L[q]_\tau^m(t+\tau) \cdot q^{(j+1)}(t) \right] dt. \quad (40) \end{aligned}$$

We now simplify the second term on the right-hand side of (40):

$$\begin{aligned} &\sum_{j=1}^m \left((\partial_{j+1} L[q]_\tau^m(t) + \partial_{j+m+2} L[q]_\tau^m(t+\tau) - \psi_1^{j-1}) \cdot q^{(j)}(t) + \psi_1^j \cdot q^{(j+1)}(t) \right) \\ &= \sum_{j=0}^{m-1} \left((\partial_{j+2} L[q]_\tau^m(t) + \partial_{j+m+3} L[q]_\tau^m(t+\tau) - \psi_1^j) \cdot q^{(j+1)}(t) + \psi_1^{j+1} \cdot q^{(j+2)}(t) \right) \\ &= \sum_{j=0}^{m-1} \left[(\partial_{j+2} L[q]_\tau^m(t) + \partial_{j+m+3} L[q]_\tau^m(t+\tau)) \cdot q^{(j+1)} \right] - \psi_1^0 \cdot \dot{q}(t) + \psi_1^m \cdot q^{(m+1)}(t). \quad (41) \end{aligned}$$

Substituting (41) into (40) and using the higher-order Euler–Lagrange equations with time delay (32), and since, by definition, $\psi_1^m = \partial_{m+2} L[q]_\tau^m(t) + \partial_{2m+3} L[q]_\tau^m(t+\tau)$ and

$$\psi_1^0 = \sum_{i=0}^m (-1)^i \frac{d^i}{dt^i} \left(\partial_{i+2} L[q]_\tau^m(t) + \partial_{i+m+3} L[q]_\tau^m(t+\tau) \right) = 0,$$

we obtain the intended result, that is,

$$\begin{aligned} \int_{t_1}^{t_2-\tau} \frac{d}{dt} \left(L[q]_\tau^m(t) - \sum_{j=1}^m \psi_1^j \cdot q^{(j)}(t) \right) dt \\ = \int_{t_1}^{t_2-\tau} \left[\partial_1 L[q]_\tau^m(t) + (\partial_{m+2} L[q]_\tau^m(t) + \partial_{2m+3} L[q]_\tau^m(t+\tau)) \cdot q^{(m+1)} \right. \\ \left. + \psi_1^0 \cdot \dot{q}(t) - \psi_1^m \cdot q^{(m+1)}(t) \right] dt = \int_{t_1}^{t_2-\tau} \partial_1 L[q]_\tau^m(t) dt. \end{aligned}$$

□

In the particular case when $m = 1$, we obtain from Theorem 18 an extension of Theorem 6 to the class of Lipschitz functions.

Corollary 19 (Nonsmooth DuBois–Reymond conditions). *If $q(\cdot) \in \text{Lip}([t_1 - \tau, t_2]; \mathbb{R}^n)$ is an extremal of functional (1), then the DuBois–Reymond conditions with time delay (6) hold true.*

Proof. For $m = 1$, condition (37) is reduced to

$$\frac{d}{dt} (L[q]_\tau(t) - \psi_1^1 \cdot \dot{q}(t)) = \partial_1 L[q]_\tau(t) \quad (42)$$

for $t_1 \leq t \leq t_2 - \tau$, and (38) to

$$\frac{d}{dt} (L[q]_\tau(t) - \psi_2^1 \cdot \dot{q}(t)) = \partial_1 L[q]_\tau(t) \quad (43)$$

for $t_2 - \tau \leq t \leq t_2$. Keeping in mind (34) and (35), we obtain

$$\psi_1^1 = \partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau) \quad (44)$$

and

$$\psi_2^1 = \partial_3 L[q]_\tau(t). \quad (45)$$

One finds the intended equalities (6) by substituting the quantities (44) and (45) into (42) and (43), respectively. \square

5.2 Higher-order Noether's symmetry theorem with time delay

Now, we generalize the Noether-type theorem proved in Section 4 to the more general case of delayed variational problems with higher-order derivatives.

Definition 20 (Invariance of (24) up to a gauge-term). *Consider the s -parameter group of infinitesimal transformations (4). Functional (24) is invariant under (4) up to the gauge-term Φ if*

$$\int_I \dot{\Phi}[q]_\tau^m(t) dt = \frac{d}{ds} \int_{\bar{t}(I)} L(\bar{t}, \bar{q}(\bar{t}), \bar{q}'(\bar{t}), \dots, \bar{q}^{(m)}(\bar{t}), \bar{q}(\bar{t} - \tau), \bar{q}'(\bar{t} - \tau), \dots, \bar{q}^{(m)}(\bar{t} - \tau)) (1 + s\dot{\eta}(t, q(t)) dt \Big|_{s=0} \quad (46)$$

for any subinterval $I \subseteq [t_1, t_2]$ and for all $q(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n)$.

Remark 21. Expressions $\dot{\Phi}$ and $\bar{q}^{(i)}$ in equation (46), $i = 1, \dots, m$, are interpreted as

$$\dot{\Phi} = \frac{d}{dt} \Phi, \quad \bar{q}' = \frac{d\bar{q}}{d\bar{t}} = \frac{d\bar{q}}{dt}, \quad \bar{q}^{(i)} = \frac{d^i \bar{q}}{d\bar{t}^i} = \frac{\frac{d}{dt} \left(\frac{d^{i-1}}{dt^{i-1}} \bar{q} \right)}{\frac{d\bar{t}}{dt}}, \quad i = 2, \dots, m. \quad (47)$$

The next lemma gives a necessary condition of invariance for functional (24).

Lemma 22 (Necessary condition of invariance for (24)). *If functional (24) is invariant up to the gauge-term Φ under the s -parameter group of infinitesimal transformations (4), then*

$$\int_{t_1}^{t_2 - \tau} \left[-\dot{\Phi}[q]_\tau^m(t) + \partial_1 L[q]_\tau^m(t) \eta(t, q) + L[q]_\tau^m(t) \dot{\eta}(t, q) + \sum_{i=0}^m (\partial_{i+2} L[q]_\tau^m(t) + \partial_{i+m+3} L[q]_\tau^m(t + \tau)) \cdot \rho^i(t) \right] dt = 0 \quad (48)$$

for $t_1 \leq t \leq t_2 - \tau$ and

$$\int_{t_2 - \tau}^{t_2} \left[-\dot{\Phi}[q]_\tau^m(t) + \partial_1 L[q]_\tau^m(t) \eta(t, q) + L[q]_\tau^m(t) \dot{\eta}(t, q) + \sum_{i=0}^m \partial_{i+2} L[q]_\tau^m(t) \cdot \rho^i(t) \right] dt = 0 \quad (49)$$

for $t_2 - \tau \leq t \leq t_2$, where

$$\begin{cases} \rho^0(t) = \xi(t, q), \\ \rho^i(t) = \frac{d}{dt} (\rho^{i-1}(t)) - q^{(i)}(t) \dot{\eta}(t, q), \quad i = 1, \dots, m. \end{cases} \quad (50)$$

Proof. Without loss of generality, we take $I = [t_1, t_2]$. Then, (46) is equivalent to

$$\int_{t_1}^{t_2} \left[-\dot{\Phi}[q]_\tau(t) + \partial_1 L[q]_\tau^m(t) \eta(t, q) + \sum_{i=0}^m \partial_{i+2} L[q]_\tau^m(t) \cdot \frac{\partial}{\partial s} \left(\frac{d^i \bar{q}}{d\bar{t}^i} \right) \Big|_{s=0} + \sum_{i=0}^m \partial_{i+m+3} L[q]_\tau^m(t) \cdot \frac{\partial}{\partial s} \left(\frac{d^i \bar{q}(\bar{t} - \tau)}{d(\bar{t} - \tau)^i} \right) \Big|_{s=0} + L[q]_\tau^m(t) \dot{\eta} \right] = 0. \quad (51)$$

Using the fact that (47) implies

$$\begin{aligned} \frac{\partial}{\partial s} \left(\frac{d\bar{q}(\bar{t})}{d\bar{t}} \right) \Big|_{s=0} &= \dot{\xi}(t, q) - \dot{q}\dot{\eta}(t, q), \\ \frac{\partial}{\partial s} \left(\frac{d^i \bar{q}(\bar{t})}{d\bar{t}^i} \right) \Big|_{s=0} &= \frac{d}{dt} \left[\frac{\partial}{\partial s} \left(\frac{d^{i-1} \bar{q}(\bar{t})}{d\bar{t}^{i-1}} \right) \Big|_{s=0} \right] - q^{(i)}(t) \dot{\eta}(t, q), \quad i = 2, \dots, m, \end{aligned}$$

then equation (51) becomes

$$\begin{aligned} \int_{t_1}^{t_2} \left[-\dot{\Phi}[q]_{\tau}^m(t) + \partial_1 L[q]_{\tau}^m(t) \eta(t, q) + L[q]_{\tau}^m(t) \dot{\eta}(t, q) \right. \\ \left. + \sum_{i=0}^m \partial_{i+2} L[q]_{\tau}^m(t) \cdot \rho^i(t) + \sum_{i=0}^m \partial_{i+m+3} L[q]_{\tau}^m(t) \cdot \rho^i(t - \tau) \right] dt = 0. \quad (52) \end{aligned}$$

Performing the linear change of variables $t = \sigma + \tau$ in the last integral of (52), and keeping in mind that $L[q]_{\tau}^m(t) \equiv 0$ on $[t_1 - \tau, t_1]$, equation (52) becomes

$$\begin{aligned} \int_{t_1}^{t_2 - \tau} \left[-\dot{\Phi}[q]_{\tau}^m(t) + \partial_1 L[q]_{\tau}^m(t) \eta(t, q) + L[q]_{\tau}^m(t) \dot{\eta}(t, q) \right. \\ \left. + \sum_{i=0}^m (\partial_{i+2} L[q]_{\tau}^m(t) + \partial_{i+m+3} L[q]_{\tau}^m(t + \tau)) \cdot \rho^i(t) \right] dt \\ + \int_{t_2 - \tau}^{t_2} \left[-\dot{\Phi}[q]_{\tau}^m(t) + \partial_1 L[q]_{\tau}^m(t) \eta(t, q) + L[q]_{\tau}^m(t) \dot{\eta}(t, q) + \sum_{i=0}^m \partial_{i+2} L[q]_{\tau}^m(t) \cdot \rho^i(t) \right] dt = 0. \quad (53) \end{aligned}$$

Equations (48) and (49) follow from the fact that (53) holds for an arbitrary $I \subseteq [t_1, t_2]$. \square

Definition 23 (Higher-order constant of motion/conservation law with time delay). *A quantity*

$$\begin{aligned} C\{q\}_{\tau}^m(t) := C\left(t, t + \tau, q(t), \dot{q}(t), \dots, q^{(m)}(t), q(t - \tau), \dot{q}(t - \tau), \dots, q^{(m)}(t - \tau), \right. \\ \left. q(t + \tau), \dot{q}(t + \tau), \dots, q^{(m)}(t + \tau)\right) \end{aligned}$$

is a higher-order constant of motion with time delay τ if

$$\frac{d}{dt} C\{q\}_{\tau}^m(t) = 0, \quad (54)$$

$t \in [t_1, t_2]$, along any $q(\cdot) \in \mathbb{W}^{m, \infty}([t_1 - \tau, t_2], \mathbb{R}^n)$ satisfying both Theorem 15 and Theorem 18. The equality (54) is then said to be a higher-order conservation law with time delay.

Theorem 24 (Higher-order Noether's symmetry theorem with time delay). *If functional (24) is invariant up to the gauge-term Φ in the sense of Definition 20, then the quantity $C\{q\}_{\tau}^m(t)$ defined by*

$$\sum_{j=1}^m \psi_1^j \cdot \rho^{j-1}(t) + \left(L[q]_{\tau}^m(t) - \sum_{j=1}^m \psi_1^j \cdot q^{(j)}(t) \right) \eta(t, q) - \Phi[q]_{\tau}^m(t) \quad (55)$$

for $t_1 \leq t \leq t_2 - \tau$ and by

$$\sum_{j=1}^m \psi_2^j \cdot \rho^{j-1}(t) + \left(L[q]_{\tau}^m(t) - \sum_{j=1}^m \psi_2^j \cdot q^{(j)}(t) \right) \eta(t, q) - \Phi[q]_{\tau}^m(t)$$

for $t_2 - \tau \leq t \leq t_2$, is a higher-order constant of motion with time delay (cf. Definition 23), where ψ_1^j and ψ_2^j are given by (34) and (35), respectively.

Proof. We prove the theorem in the interval $t_1 \leq t \leq t_2 - \tau$. The proof is similar in the interval $t_2 - \tau \leq t \leq t_2$. Equation (55) follows by direct calculations:

$$\begin{aligned}
0 &= \int_{t_1}^{t_2-\tau} \frac{d}{dt} \left[\psi_1^1 \cdot \rho^0 + \sum_{j=2}^m \psi_1^j \cdot \rho^{j-1}(t) + \left(L[q]_\tau^m(t) - \sum_{j=1}^m \psi_1^j \cdot q^{(j)}(t) \right) \eta(t, q) - \Phi[q]_\tau^m(t) \right] dt \\
&= \int_{t_1}^{t_2-\tau} \left[-\dot{\Phi}[q]_\tau^m(t) + \rho^0(t) \cdot \frac{d}{dt} \psi_1^1 + \psi_1^1 \cdot \frac{d}{dt} \rho^0(t) + \sum_{j=2}^m \left(\rho^{j-1}(t) \cdot \frac{d}{dt} \psi_1^j + \psi_1^j \cdot \frac{d}{dt} \rho^{j-1}(t) \right) \right. \\
&\quad \left. + \eta(t, q) \frac{d}{dt} \left(L[q]_\tau^m(t) - \sum_{j=1}^m \psi_1^j \cdot q^{(j)}(t) \right) + \left(L[q]_\tau^m(t) - \sum_{j=1}^m \psi_1^j \cdot q^{(j)}(t) \right) \dot{\eta}(t, q) \right] dt. \tag{56}
\end{aligned}$$

Using the Euler–Lagrange equation (32), the DuBois–Reymond condition (37), and relations (36) and (50) in (56), we obtain:

$$\begin{aligned}
&\int_{t_1}^{t_2-\tau} \left[-\dot{\Phi}[q]_\tau^m(t) + (\partial_2 L[q]_\tau^m(t) + \partial_{m+3} L[q]_\tau^m(t + \tau)) \cdot \xi(t, q) + \psi_1^1 \cdot (\rho^1(t) + \dot{q}(t) \dot{\tau}(t, q)) \right. \\
&\quad \left. + \sum_{j=2}^m \left[(\partial_{j+1} L[q]_\tau^m(t) + \partial_{j+m+2} L[q]_\tau^m(t + \tau) - \psi_1^{j-1}) \cdot \rho^{j-1}(t) + \psi_1^j \cdot (\rho^j(t) + q^{(j)}(t) \dot{\tau}(t, q)) \right] \right. \\
&\quad \left. + \partial_1 L[q]_\tau^m(t) \eta(t, q) + \left(L[q]_\tau^m(t) - \sum_{j=1}^m \psi_1^j \cdot q^{(j)}(t) \right) \dot{\eta}(t, q) \right] dt \\
&= \int_{t_1}^{t_2-\tau} \left[\partial_1 L[q]_\tau^m(t) \eta(t, q) + L[q]_\tau^m(t) \dot{\eta}(t, q) + (\partial_2 L[q]_\tau^m(t) + \partial_{m+3} L[q]_\tau^m(t + \tau)) \cdot \xi(t, q) \right. \\
&\quad \left. + \psi_1^1 \cdot (\rho^1(t) + \dot{q}(t) \dot{\eta}(t, q)) - \psi_1^1 \cdot \rho^1(t) - \psi_1^1 \cdot \dot{q}(t) \dot{\eta}(t, q) + \psi_1^m \cdot \rho^m(t) \right. \\
&\quad \left. + \sum_{j=2}^m (\partial_{j+1} L[q]_\tau^m(t) + \partial_{j+m+2} L[q]_\tau^m(t + \tau)) \cdot \rho^{j-1}(t) - \dot{\Phi}[q]_\tau^m(t) \right] dt = 0. \tag{57}
\end{aligned}$$

Simplification of (57) leads to the necessary condition of invariance (48). \square

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